Journal of Statistical Physics, Vol. 66, Nos. 3/4, 1992

Phase Transitions in Thermodynamics of a Local Lyapunov Exponent for Fully-Developed Chaotic Systems

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Received December 10, 1990; final August 13, 1991

Fluctuations in the divergence of nearby orbits are studied at a crisis point of chaos. A statistical-thermodynamic method for the description of the fluctuations is developed by using symbolic dynamics, which can explicitly write a relation between a fluctuation and reference orbit. The thermodynamics (the free energy and entropy) is exactly analyzed on a nonhyperbolic attractor of maps conjugate to the map: $u \to u/a$ for $0 \leqslant u < a$ and $u \to (1-u)/(1-a)$ for $a \leqslant u \leqslant 1$. Te free energy has discontinuities in its slope. The entropy is directly calculated from the partition function. Then, it becomes clear that the collision of a chaotic attractor with a particular fixed point yields a singular local structure in the distribution of fluctuations. The existence of first-order phase transitions depends on the asymmetry of a map. It is shown that each of the coexisting states at the phase transition points is realized with the same probability in the thermodynamic limit.

KEY WORDS: Fully developed chaos; local Lyapunov exponent; thermodynamics; exact solutions; first-order phase transitions; entropy; coexisting states.

1. INTRODUCTION

Recently, fluctuations in the divergence of nearby orbits have been extensively studied to describe the dynamical features of chaos.[1-11] It has been shown that at a bifurcation point, such as a crisis point and a saddle-node bifurcation point of intermittent chaos, the distribution of these fluctuations takes anomalous forms. In this paper, the fluctuations on attractors

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0022-4715/92/0200-0727$06.50/0

of fully developed chaos[12] will be studied in the statistical-thermodynamic formalism,[13] and anomalous behaviors of the fluctuations will be analyzed in detail.

A local Lyapunov exponent Λ is defined as a divergence rate of infinitely neighboring orbits along a reference orbit for a finite time n.[14–16] A statistical-thermodynamic method[15–19] for the description of fluctuations of a local Lyapunov exponent has been introduced, using a partition function. Then, the anomalous behavior of the fluctuations can be characterized by nonanalyticity of the free energy in the thermodynamic limit $n \to \infty$.[4–10] The logistic map, $f(x) = 4x(1-x)$, gives the simplest example that the free energy having discontinuities in its slope can be solved exactly.[8,15] There are other thermodynamics, such as generalized dimensions[17,20] and generalized entropies,[21] for characterizing chaos. For the logistic map, a phase transition occurs in the thermodynamics of a scaling index, i.e., in the generalized dimensions.[15,22] Is there any relation between these phase transitions?[8,16] The distribution of local Lyapunov exponents depends on that of initial values of the reference orbits. Usually, the probability measure of an initial value is given by a natural invariant measure. I introduce generalized thermodynamics of a local Lyapunov exponent, using weighted measures instead of the natural measure. Then, a relation between the phase transitions in the thermodynamics of a scaling index and of a local Lyapunov exponent becomes clear in the generalized thermodynamics.

Large deviations of fluctuations of a local Lyapunov exponent can be described by the entropy.[23,24] Anomalous behavior of the fluctuations may be characterized by linear slopes of the entropy. The linear slopes imply that the free energy has discontinuities in its slope, i.e., the occurrence of first-order phase transitions. Usually, the entropy is given as the Legendre transform of the free energy.[15,19] Discontinuities in the slope of the free energy then lead to linear slopes of the entropy, because the Legendre transform is convex. One must notice that the entropy may be different from the Legendre transform when the free energy has discontinuities.[16] The distribution of fluctuations can be directly obtained from experiments, while the free energy is obtained indirectly.[4–6,25] The distribution of the fluctuations, i.e., the entropy, must be obtained from direct calculations, not from the Legendre transformation of the free energy, when the free energy has the discontinuities. In this paper, the entropy will be directly calculated from the partition function. The calculations are performed by using symbolic dynamics.[16,26] Since the symbolic dynamics describes the relation of a local Lyapunov exponent to a reference orbit explicitly, it is possible to get detailed information about typical reference orbits for each value of a local Lyapunov exponent. Then, the linear slopes of the entropy for fully developed chaos can be explained in relation to the typical reference orbits.

For the logistic map $f(x)=4x(1-x)$, the entropy of a local Lyapunov exponent (w.r.t. the natural measure) is[8,15]

$$s(\Lambda)=\begin{cases}\Lambda-\ln 2 & \text{for} \quad \Lambda\leqslant \ln 2\\ \ln 2-\Lambda & \text{for} \quad \ln 2<\Lambda\leqslant 2\ln 2\\ -\infty & \text{for} \quad \Lambda\geqslant 2\ln 2\end{cases} \tag{1.1}$$

Since the logistic map is conjugate to the symmetrical tent map $g(x)=1-|2x-1|$, almost all of the fluctuations are degenerate with ln 2. The fixed point $x=0$ has $\Lambda=2\ln 2$. Bohr and Jensen[8] studied the fully developed chaos of asymmetrical maps, discussing effects of the asymmetric perturbation to the singular local structures of fluctuations given by (1.1). They found that a large perturbation of the asymmetry may lead to qualitative changes of the singular local structures, while a small perturbation resolves the degeneracy of ln 2 but gives no qualitative change. Their work was done in poor numerical calculations, and a uniform measure not equal to the natural measure was used. In this paper, the influence of an asymmetrical perturbation on the thermodynamics of a local Lyapunov exponent will be analytically investigated by using maps conjugate to an asymmetrical tent map.

Section 2 gives maps conjugate to an asymmetrical tent map and constructs symbolic dynamics by using a dynamical partition. The slope λ_0 of the asymmetrical tent map at the fixed point $x=0$ measures the degree of asymmetry for the conjugating maps. In Section 3, the free energy of a local Lyapunov exponent is calculated and shows qualitatively different non-analytic behaviors dependent on λ_0. Using weighted measures for an initial value of a reference orbit, the free energy is formulated in a generalized form. A relation between the thermodynamics of a scaling index and of a local Lyapunov exponent, especially the occurrence of first-order phase transitions, becomes clear. In Section 4, the entropy is directly calculated from the partition function. Typical reference orbits are given for each value of a local Lyapunov exponent. Linear slopes of the entropy are explained in intuitive discussions. Section 5 discusses probabilities of coexisting states at a first-order phase transition point. I show that each of the coexisting states is realized with the same probability in the thermodynamic limit. Section 5 gives summary also. Details of the calculations are performed in appendices.

2. MAPS

Let us consider a map of the unit interval $I \equiv [0, 1]$ to itself:

$$g(u) = \begin{cases} \lambda_0 u & \text{for} \quad 0 \leqslant u < a \\ \lambda_1(1-u) & \text{for} \quad a \leqslant u \leqslant 1 \end{cases} \tag{2.1}$$

where $\lambda_0 = 1/a$ and $\lambda_1 = 1/(1-a)$. An attractor of g is the entire interval I. By using g, the interval I is partitioned into small intervals labeled by binary strings:

$$J_u(\sigma_1 \sigma_2 \cdots \sigma_n) \equiv J_u(\sigma_1) \cap g^{-1}(J_u(\sigma_2 \cdots \sigma_n)) \tag{2.2a}$$

for $n = 2, 3, 4,...,$ and

$$J_u(\sigma_1) \equiv \begin{cases} [0, a) & \text{for} \quad \sigma_1 = 0 \\ [a, 1] & \text{for} \quad \sigma_1 = 1 \end{cases} \tag{2.2b}$$

where $\sigma_j \in \{0, 1\}$. The length of the interval $J_u(\sigma_1 \sigma_2 \cdots \sigma_n)$ is

$$|J_u(\sigma_1 \sigma_2 \cdots \sigma_n)| = \prod_{j=1}^{n} [1/\lambda(\sigma_j)] \tag{2.3}$$

where $\lambda(\sigma) \equiv (1-\sigma)\lambda_0 + \sigma\lambda_1$. The natural invariant measure for g is given by the length of an interval:

$$P_g(\sigma_1 \sigma_2 \cdots \sigma_n) = |J_u(\sigma_1 \sigma_2 \cdots \sigma_n)| \tag{2.4}$$

$u(\sigma_1 \sigma_2 \cdots \sigma_n)$ denotes the minimum point in the interval $J_u(\sigma_1 \sigma_2 \cdots \sigma_n)$. Then, we have

$$\begin{aligned} u(\sigma_1 \sigma_2 \cdots \sigma_n) &\equiv \min\{u \mid u \in J_u(\sigma_1 \sigma_2 \cdots \sigma_n)\} \\ &= \sum_{j=1}^{n} |J_u(\sigma_1 \cdots \sigma_{j-1} \bar{\sigma}_j)| \, \Theta\left(\sum_{k=1}^{j} \sigma_k \right) \end{aligned} \tag{2.5}$$

where $\bar{\sigma} \equiv 1 - \sigma$ and the integer function Θ is

$$\Theta(i) = \begin{cases} 0 & \text{for} \quad i = \text{even} \\ 1 & \text{for} \quad i = \text{odd} \end{cases}$$

Let us now consider a map conjugate to the asymmetrical tent map (Fig. 1),

$$f = \varphi \circ g \circ \varphi^{-1} \tag{2.6}$$

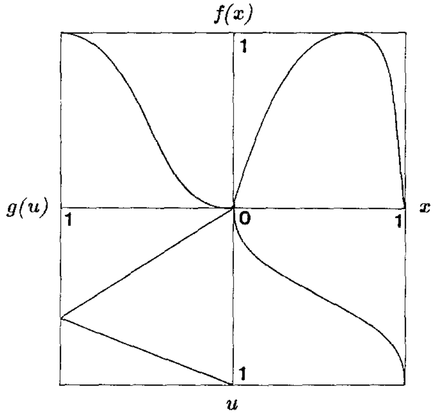


Fig. 1. An asymmetrical tent map $g(u)$ and conjugating map $f(x)$. The conjugacy is $x=\varphi(u)=u^2/[u^2+(1-u)^2]$.

The conjugacy φ is assumed to be a smooth and strictly monotone increasing function on the open interval (0, 1) with $\varphi(0)=0$ and $\varphi(1)=1$. We consider the dynamical partition consisting of the 2^n intervals labeled by the binary strings,

$$J(\sigma_1\sigma_2\cdots\sigma_n)=\{x\,|\,x=\varphi(u),\ u\in J_u(\sigma_1\sigma_2\cdots\sigma_n)\} \tag{2.7}$$

From the mean value theorem, the length of $J(\sigma_1\sigma_2\cdots\sigma_n)$ can be written as

$$|J(\sigma_1\sigma_2\cdots\sigma_n)|=|J_u(\sigma_1\sigma_2\cdots\sigma_n)|\ \varphi'(\tilde{u}(\sigma_1\sigma_2\cdots\sigma_n)) \tag{2.8a}$$

where φ' is the derivative of φ and the constant $\tilde{u}$ satisfies the inequality

$$u(\sigma_1\sigma_2\cdots\sigma_n)<\tilde{u}(\sigma_1\sigma_2\cdots\sigma_n)<u(\sigma_1\sigma_2\cdots\sigma_n)+|J_u(\sigma_1\sigma_2\cdots\sigma_n)| \tag{2.8b}$$

The probability on $J(\sigma_1\sigma_2\cdots\sigma_n)$ is

$$P_f(\sigma_1\sigma_2\cdots\sigma_n)=P_g(\sigma_1\sigma_2\cdots\sigma_n) \tag{2.9}$$

If $\varphi'(u)$ is positive and finite on I, then $|J(\sigma_1\sigma_2\cdots\sigma_n)|$ and $|J_u(\sigma_1\sigma_2\cdots\sigma_n)|$ show the same asymptotic behaviors in the limit $n\to\infty$. Hence, the thermodynamics of a local Lyapunov exponent, given in the next section, is invariant under the conjugation of this type. We consider the case that φ has the same singularity at $u=0$ and $u=1$:

$$\varphi(u)\simeq\begin{cases}B_1u^z & \text{for}\quad 0\leqslant u\ll 1\\ 1-B_2(1-u)^z & \text{for}\quad 0\leqslant 1-u\ll 1\end{cases} \tag{2.10}$$

where $z>0$ and $z\neq 1$. Here B_1 and B_2 are positive constants of $O(1)$. Therefore, the conjugating function f has a maximum of the order z at $x=c\equiv\varphi(a)$,

$$f(x)\simeq\begin{cases}1-B_2[a\varphi'(a)]^{-z}(c-x)^z & \text{for}\quad 0\leqslant c-x\ll 1\\ 1-B_2[(1-a)\,\varphi'(a)]^{-z}(x-c)^z & \text{for}\quad 0\leqslant x-c\ll 1\end{cases} \tag{2.11}$$

Note that the slope of f at $x=0$ is λ_0^z.

3. FREE ENERGY AND FIRST-ORDER PHASE TRANSITIONS

A local Lyapunov exponent of a reference orbit $\{x_0, x_1=f(x_0),..., x_n=f^n(x_0)\}$ is given by

$$\Lambda(x_0;n)=\frac{1}{n}\sum_{j=1}^{n}\ln|f'(x_{j-1})| \tag{3.1}$$

For $u_0\in J_u(\sigma_1\sigma_2\cdots\sigma_n)$, we have

$$\Lambda(u_0;n)=\frac{1}{n}\sum_{j=1}^{n}\ln\lambda(\sigma_j) \tag{3.2}$$

However, for different x_0 and $x_0'\in J(\sigma_1\sigma_2\cdots\sigma_n)$ on the attractor of f, $\Lambda(x_0;n)$ generally has a different value from $\Lambda(x_0';n)$. Let us consider an expansion rate of a small interval, instead of (3.1)[15,16]:

$$\tilde{\Lambda}(\sigma_1\sigma_2\cdots\sigma_N;n)\equiv(1/n)\ln\{|J(\sigma_{n+1}\cdots\sigma_N)|/|J(\sigma_1\sigma_2\cdots\sigma_N)|\} \tag{3.3}$$

The local Lyapunov exponent (3.1) is obtained from (3.3) in the limit $N\to\infty$.

The statistical thermodynamics of $\tilde{\Lambda}$ is introduced with the Gibbs ensemble

$$\rho(\sigma_1\sigma_2\cdots\sigma_N;q,t;n)\equiv[P_f(\sigma_1\sigma_2\cdots\sigma_N)]^q\exp\{-nt\tilde{\Lambda}(\sigma_1\sigma_2\cdots\sigma_N;n)\}/\Xi_N(q,t;n) \tag{3.4}$$

The partition function $\Xi_N(q,t;n)$ is given by

$$\Xi_N(q,t;n)\equiv\sum_{\sigma_1\sigma_2\cdots\sigma_N}[P_f(\sigma_1\sigma_2\cdots\sigma_N)]^q\exp\{-nt\tilde{\Lambda}(\sigma_1\sigma_2\cdots\sigma_N;n)\} \tag{3.5}$$

where the summation is taken over 2^N configurations of a binary string. On the assumption of convergence in the limit $n\to\infty$ with $N/n=r$ fixed, the free energy is defined by

$$G\left(q,t;\frac{N}{n}=r\right)\equiv-\lim_{n\to\infty}\frac{1}{n}\ln\Xi_N(q,t;n) \tag{3.6}$$

$G(q, t; r)$ agrees with the free energy of a scaling index and size index of a box at $r=1$,[16] with that of Bohr and Jensen at $q=0$ and $r=1$,[8] and with that of a local Lyapunov exponent at $q=1$ and $r=\infty$. For the asymmetrical tent map, it follows that

$$G(q, t; r) = -\ln A(q+t) - (r-1) \ln A(q) \tag{3.7a}$$

where

$$A(q) \equiv (1/\lambda_0)^q + (1/\lambda_1)^q \tag{3.7b}$$

Note that (3.7) is analytic for q and t.

For the conjugating function, the partition function (3.5) can be written, inserting (2.8a) into (3.3), (2.4) into (2.9), and using (2.3), as

$$\begin{aligned} \Xi_N(q, t; n) = &\sum_{\sigma_1 \sigma_2 \cdots \sigma_N} \left[\frac{\varphi'(\tilde{u}(\sigma_1 \sigma_2 \cdots \sigma_N))}{\varphi'(\tilde{u}(\sigma_{n+1} \cdots \sigma_N))} \right]^t \\ &\times \prod_{j=1}^{n} \left[\frac{1}{\lambda(\sigma_j)} \right]^{q+t} \prod_{k=n+1}^{N} \left[\frac{1}{\lambda(\sigma_k)} \right]^q \end{aligned} \tag{3.8}$$

If $0 < \varphi'(u) < +\infty$ for all $u \in I$, then the free energy for f becomes (3.7). In the case (2.10), it turns out from Appendix A that

$$\begin{aligned} \Xi_N(q, t; n) \sim &\frac{[A(q+t)]^{n-1} - (1/\lambda_0)^{(q+zt)(n-1)}}{A(q+t) - (1/\lambda_0)^{q+zt}} \\ &\times \frac{[A(q)]^{N-n-1} - (1/\lambda_0)^{(q+t-zt)(N-n-1)}}{A(q) - (1/\lambda_0)^{q+t-zt}} \end{aligned} \tag{3.9}$$

for sufficiently large N and n. At $r=1$, the free energy is

$$G(q, t; r=1) = \begin{cases} -\ln A(q+t) & \text{if} \quad -\ln A(q+t) < (q+zt) \ln \lambda_0 \\ (q+zt) \ln \lambda_0 & \text{if} \quad -\ln A(q+t) > (q+zt) \ln \lambda_0 \end{cases} \tag{3.10}$$

Note that a solution of the equation

$$G(q, t = -\tau(q); r=1) = 0$$

gives the free energy of a scaling index[15,16]

$$\tau(q) = \begin{cases} q/z & \text{for} \quad q \geqslant z/(z-1) \quad \text{if} \quad z > 1 \\ & \text{and for} \quad q < z/(z-1) \quad \text{if} \quad 0 < z < 1 \\ q-1 & \text{for} \quad q < z/(z-1) \quad \text{if} \quad z > 1 \\ & \text{and for} \quad q \geqslant z/(z-1) \quad \text{if} \quad 0 < z < 1 \end{cases} \tag{3.11}$$

For $r > 1$, the free energy is given by

$$G(q, t; r > 1) = \min\{G_A, G_B, G_C\} \tag{3.12}$$

where G_A, G_B, and G_C are the free energies in the phases A, B, and C, respectively:

$$G_A(q, t; r > 1) \equiv -\ln A(q+t) + (r-1)(q+t-zt) \ln \lambda_0 \tag{3.13a}$$

$$G_B(q, t; r > 1) \equiv -\ln A(q+t) - (r-1) \ln A(q) \tag{3.13b}$$

$$G_C(q, t; r > 1) \equiv (q+zt) \ln \lambda_0 - (r-1) \ln A(q) \tag{3.13c}$$

The singularity of the derivative of f at $x = c$, i.e., $|f'(x)| = 0$ if $z > 1$ and $= +\infty$ if $0 < z < 1$ in the limit $x \to c$, brings out the phase A. The free energy G_B coincides with (3.7), so that the phase B is called the hyperbolic phase. The phase C is brought out by the singularity of the probability measure at $x = 0$, called the fixed point phase. Thus, the free energy (3.12) has nonanalytic points.

We now study the phase diagram in the (λ_0, t, q) plane. The boundary between the phases A and B is given by

$$\lambda_0^{(z-1)t} = 1 + (\lambda_0/\lambda_1)^q \tag{3.14}$$

A solution of (3.14), $t = t(q, \lambda_0)$, is positive if $z > 1$ and negative if $0 < z < 1$. The boundary between the phases B and C is given by

$$(z-1)t \ln \lambda_0 + \ln[1 + (\lambda_0/\lambda_1)^{q+t}] = 0 \tag{3.15}$$

For $z > 1$ and $q = 1$, we get the following:

1. If (3.15) has real solutions, they are negative.
2. Equation (3.15) has a unique real solution for $\lambda_0 \geq \lambda_*$, two real solutions for $\lambda_* > \lambda_0 \geq \lambda_c$, and no real solution for $\lambda_c > \lambda_0 > 1$. At $\lambda_0 = \lambda_c$, the solutions become degenerate. [The definitions of λ_* and λ_c are given in Appendix B, where (3.15) is analyzed in detail.]

Figure 2 displays the phase diagrams on the plane $(\lambda_0, t, q = 1)$ for $z > 1$ and for $0 < z < 1$, respectively. When f is conjugate to the symmetrical tent map, i.e., $\lambda_0 = \lambda_1 = 2$, G_A and G_B are linear functions of q and t. As t changes in the case $z > 1$, $G(q = 1, t; r > 1)$ shows different types of phase changes, depending on λ_0: There are two phase transitions for $\lambda_0 \geq \lambda_*$ (along the line SS' in Fig. 2a), three phase transitions for $\lambda_* > \lambda_0 > \lambda_c$

(along the line RR'), and only one phase transition for $\lambda_c > \lambda_0 > 1$ (along the line QQ'). (See Fig. 3 also.) Bohr and Jensen have numerically obtained similar results for the fully developed map $f(x) = Ax(1-x)(1+\gamma x)$.[8] Their aim is to study $G(q=0, t; r=1)$, which does not have the phase A. However, they observed the phase transition between the phases A and B for a small value of the asymmetry γ. This discrepancy may come from the fact that they actually calculated the maximum eigenvalue and its eigen-

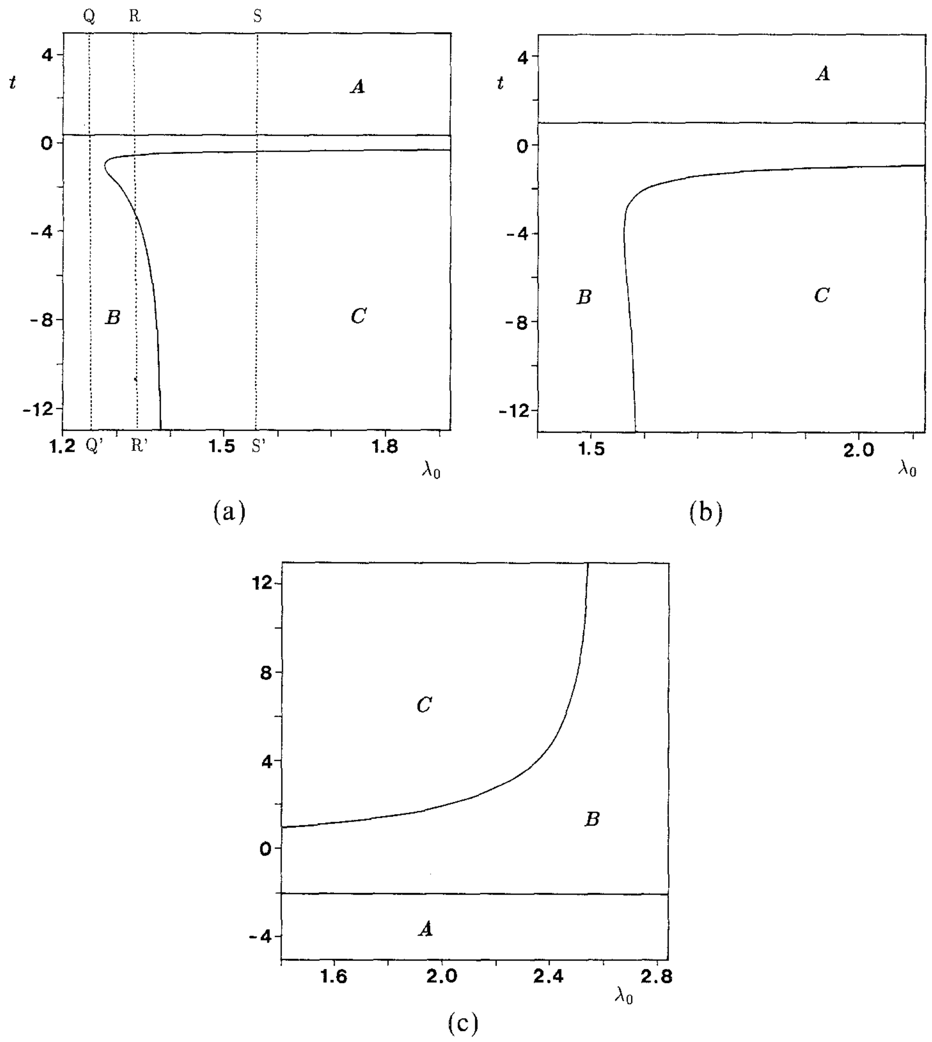


Fig. 2. Phase diagram on the plane (λ_0, t, $q=1$), for (a) $z=4$, (b) $z=2$, and (c) $z=1/2$. The phase boundary of A and B is given by the line $t=1/(z-1)$.

vector of the scaling function $\exp[-\tilde{\Lambda}(\sigma_1\sigma_2\cdots\sigma_N; n=1)]$,[26] which is different from $G(q=0, t; r=1)$. Figure 4 gives $G(q, t; r=1)$ as a function of q and t for $\lambda_* > \lambda_0 > \lambda_c$ as well as the free energy of a scaling index. The phase diagram at $q=0$ is qualitatively different from that at $q=1$. One can also find there a relation between the phase transitions in both the thermodynamics of a scaling index and of a local Lyapunov exponent. The phase transitions, occurring on the changes of t with q fixed, are of the first

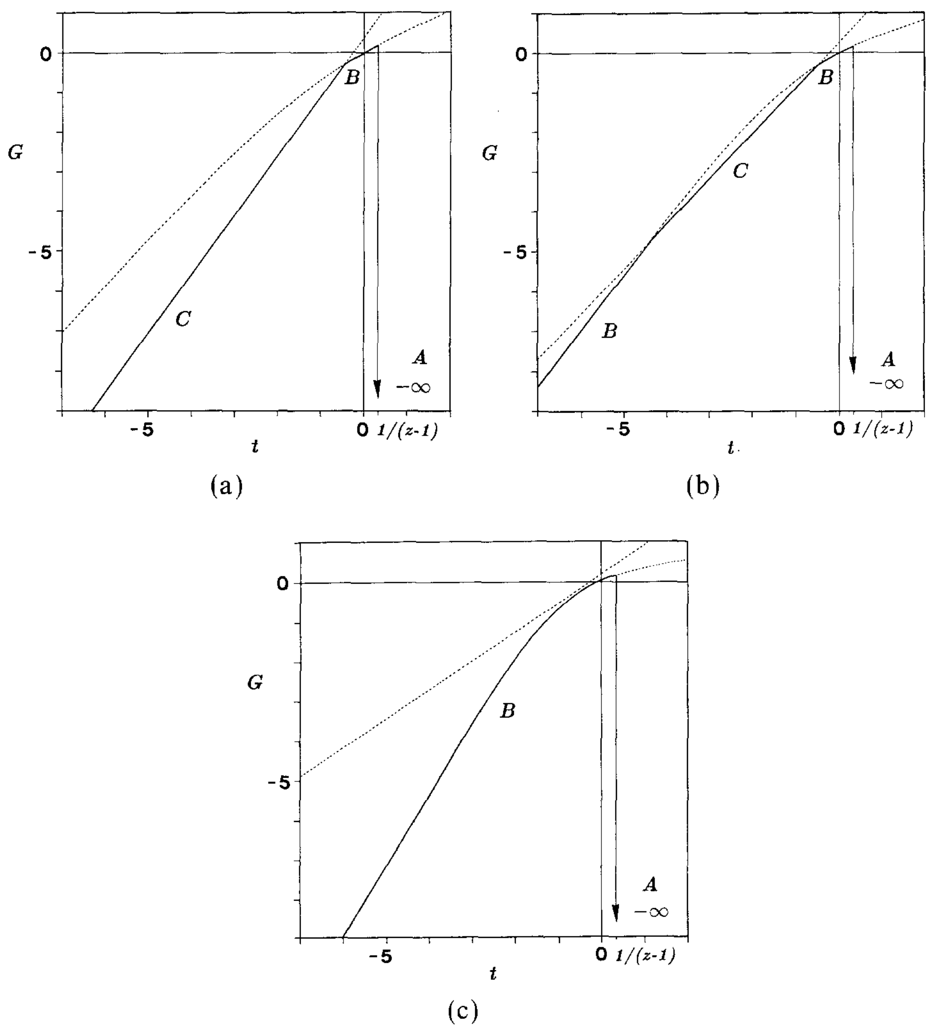


Fig. 3. Free energy $G(q=1, t; r=\infty)$ for $\lambda_0 > \lambda_*$, $\lambda_* > \lambda_0 > \lambda_c$, and $\lambda_c > \lambda_0 > 1$. Parameter values are $z=4$ and $\lambda_0=$ (a) 1.45, (b) 1.33, and (c) 1.2.

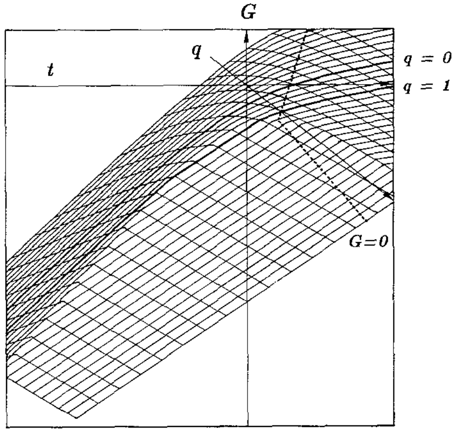


Fig. 4. Free energy $G(q, t; r = 1)$. Parameter values are $z = 4$ and $\lambda_0 = 1.3$, which satisfies $\lambda_* > \lambda_0 > \lambda_c(q = 1)$. The free energy of Bohr and Jensen is given by the heavy line $q = 0$. The heavy line $q = 1$ coincides with the free energy of a local Lyapunov exponent for $t < 1/(z-1)$. The dashed line depicts $G(q, t; r = 1) = 0$, which gives $t = -\tau(q)$.

order, except of a particular $q = q_c(\lambda_0)$. Note that a finite-size scaling law for these first-order phase transitions holds in the normal form.[16,27] The scaling exponent $\zeta = 1$ is different from Bohr and Jensen's numerical results.[8]

4. ENTROPY FUNCTION AND ITS LINEAR SLOPES

We study large deviations of the fluctuations of a local Lyapunov exponent.[4] Assume that the probability of an initial value of a reference orbit on the interval $J(\sigma_1\sigma_2\cdots\sigma_N)$ is given by

$$[P_f(\sigma_1\sigma_2\cdots\sigma_N)]^q/\Xi_N(q, t=0; n) \tag{4.1}$$

which coincides with the natural invariant measure at $q = 1$. Now, $W_N(\tilde{\Lambda}; n, q)\, d\tilde{\Lambda}$ is the probability that $\tilde{\Lambda}(\sigma_1\sigma_2\cdots\sigma_N; n)$ takes a value on the interval $[\tilde{\Lambda}, \tilde{\Lambda} + d\tilde{\Lambda}]$. On the assumption of the existence of a limit, the entropy function of $\tilde{\Lambda}$ is defined by

$$s(\tilde{\Lambda}; q, r) \equiv \lim_{n\to\infty} (1/n) \ln W_N(\tilde{\Lambda}; n, q) \qquad \text{with} \quad N/m = r \text{ fixed} \tag{4.2}$$

The partition function (3.5) can be written as

$$\Xi_N(q, t; n) \sim \Xi_N(q, t=0; n) \int d\tilde{\Lambda} \exp\{n[s(\tilde{\Lambda}; q, r) - t\tilde{\Lambda}]\} \tag{4.3}$$

for sufficiently large n. Therefore, we have

$$\Delta G \equiv G(q, t; r) - G(q, t=0; r) = \min_{\Lambda}\{t\Lambda - s(\Lambda; q, r)\}$$

Using Stirling approximations in (A.5) and replacing the summations with integrations, we get

$$\tilde{\Lambda} = (z-1)(\eta - \zeta) \ln \lambda_0 + \ln \lambda_0 - \xi \ln(\lambda_0/\lambda_1) \tag{4.4}$$

$$\exp\{ns(\tilde{\Lambda}; q, r)\} \sim \frac{n^3}{(z-1)\ln \lambda_0} \iint_{D(\tilde{\Lambda}; q, r)} d\xi\, d\eta \exp\{-nF(\xi, \eta; \tilde{\Lambda}, q)\} \tag{4.5}$$

where

$$\begin{aligned} F(\xi, \eta; \tilde{\Lambda}, q) \equiv{} & (1-\xi-\eta)\ln(1-\xi-\eta) - (1-\eta)\ln(1-\eta) \\ & + \xi \ln \xi - q\xi \ln(\lambda_0/\lambda_1) \\ & + \{\eta - [\tilde{\Lambda} - z \ln \lambda_0 + \xi \ln(\lambda_0/\lambda_1)]/(z-1)\ln \lambda_0\} \\ & \times \ln[1 + (\lambda_0/\lambda_1)^q] \end{aligned} \tag{4.6}$$

and the region of integration, denoted by D, is dependent on $\tilde{\Lambda}$. For simplicity, we consider the case $z > 1$ and $r \to \infty$. Figures 5–7 show $D(\tilde{\Lambda}; q, r=\infty)$ and a minimum point Q of $F(\xi, \eta; \tilde{\Lambda}, q)$ on D for $2 > \lambda_0 > \lambda_*$, for $\lambda_* > \lambda_0 > \lambda_c$, and for $\lambda_c > \lambda_0 > 1$, respectively. As maximum value approximations can be used to the integral (4.5), the following results are obtained (Fig. 8).

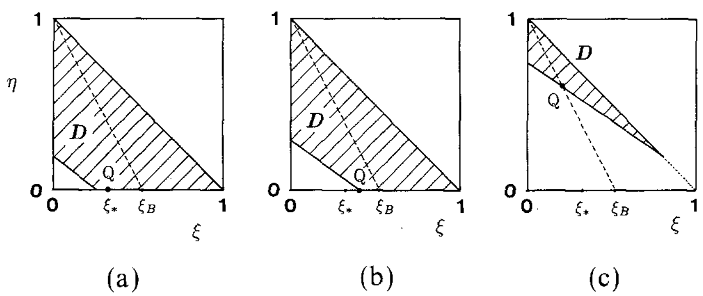


Fig. 5. A minimum point Q of $F(\xi, \eta; \Lambda, q)$ on $D = D(\Lambda; q, r=\infty)$ when $\lambda_* < \lambda_0 < 2$, for (a) $\Lambda \leqslant \Lambda_a$, (b) $\Lambda_a < \Lambda \leqslant \Lambda_b$, and (c) $\Lambda_b < \Lambda \leqslant z \ln \lambda_0$. Here ξ_B denotes $\xi_0(\Lambda_B)$.

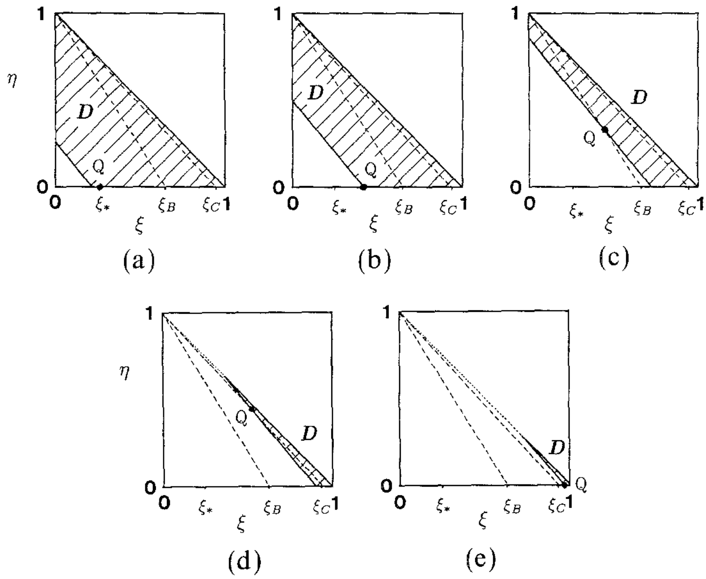


Fig. 6. A minimum point Q of $F(\xi, \eta; \Lambda, q)$ on $D = D(\Lambda; q, r = \infty)$ when $\lambda_c < \lambda_0 < \lambda_*$, for (a) $\Lambda \leqslant \Lambda_a$, (b) $\Lambda_a < \Lambda \leqslant \Lambda_b$, (c) $\Lambda_b < \Lambda \leqslant z \ln \lambda_0$, (d) $z \ln \lambda_0 < \Lambda \leqslant \Lambda_c$, and (e) $\Lambda_c < \Lambda \leqslant \ln \lambda_1$. Here ξ_C denotes $\xi_0(\Lambda_c)$.

If $\lambda_0 > \lambda_*$ and $\lambda_0 \neq \lambda_1$, then

$$s(\Lambda; q, r = \infty) = \begin{cases} s_h(\Lambda_a; q) + (\Lambda - \Lambda_a)\, t^*(\Lambda_a; q) & \text{for} \quad \Lambda \leqslant \Lambda_a \\ s_h(\Lambda; q) & \text{for} \quad \Lambda_a < \Lambda \leqslant \Lambda_b \\ s_h(\Lambda_b; q) + (\Lambda - \Lambda_b)\, t^*(\Lambda_b; q) & \text{for} \quad \Lambda_b < \Lambda \leqslant z \ln \lambda_0 \\ -\infty & \text{for} \quad \Lambda > z \ln \lambda_0 \end{cases} \tag{4.7a}$$

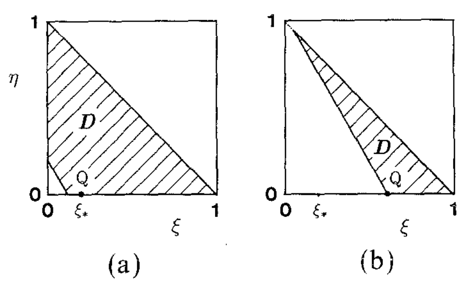
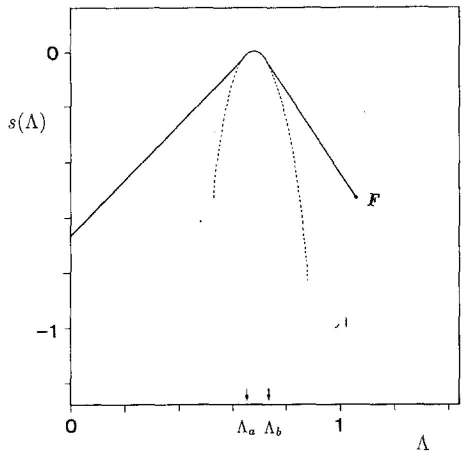
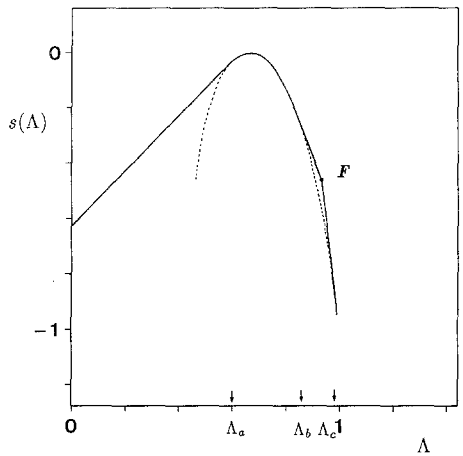


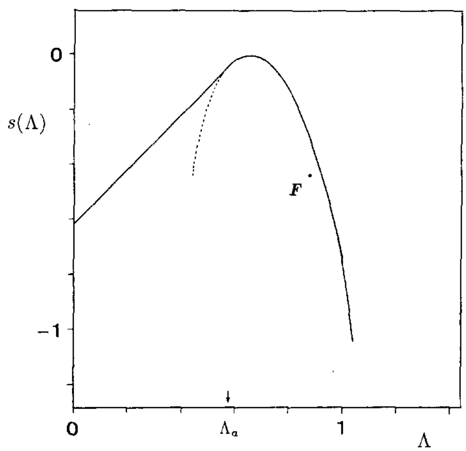
Fig. 7. A minimum point Q of $F(\xi, \eta; \Lambda, q)$ on $D = D(\Lambda; q, r = \infty)$ when $1 < \lambda_0 < \lambda_c$, for (a) $\Lambda \leqslant \Lambda_a$ and (b) $\Lambda_a < \Lambda \leqslant \ln \lambda_1$.



(a)



(b)



(c)

Fig. 8. Entropy function $s(\Lambda; q=1, r=\infty)$ for $\lambda_* < \lambda_0 < 2$, $\lambda_c < \lambda_0 < \lambda_*$, and $1 < \lambda_0 < \lambda_c$. The dashed line denotes the entropy function $s_h(\Lambda, q=1)$ for the asymmetrical tent map. The straight lines are tangent to the curve $s_h(\Lambda, q=1)$ at $\Lambda = \Lambda_a$, $\Lambda = \Lambda_b$, and $\Lambda = \Lambda_c$, respectively. The term coming from the fixed point $x=0$ is denoted by F. Parameter values are $z=2$ and $\lambda_0 =$ (a) 1.7, (b) 1.59, and (c) 1.55.

If $\lambda_* > \lambda_0 > \lambda_c$, then

$$s(\Lambda; q, r=\infty) = \begin{cases} s_h(\Lambda_a; q) + (\Lambda - \Lambda_a)\, t^*(\Lambda_a; q) & \text{for } \Lambda \leqslant \Lambda_a \\ s_h(\Lambda; q) & \text{for } \Lambda_a < \Lambda \leqslant \Lambda_b \text{ or } \Lambda_c < \Lambda \leqslant \ln \lambda_1 \\ s_h(\Lambda_b; q) + (\Lambda - \Lambda_b)\, t^*(\Lambda_b; q) & \text{for } \Lambda_b < \Lambda \leqslant z \ln \lambda_0 \\ s_h(\Lambda_c; q) + (\Lambda - \Lambda_c)\, t^*(\Lambda_c; q) & \text{for } z \ln \lambda_0 < \Lambda \leqslant \Lambda_c \\ -\infty & \text{for } \Lambda > \ln \lambda_1 \end{cases} \tag{4.7b}$$

If $\lambda_c > \lambda_0 > 1$, then

$$s(\Lambda; q, r=\infty) = \begin{cases} s_h(\Lambda_a; q) + (\Lambda - \Lambda_a)\, t^*(\Lambda_a; q) & \text{for } \Lambda \leqslant \Lambda_a \\ s_h(\Lambda; q) & \text{for } \Lambda_a < \Lambda \leqslant \ln \lambda_1 \\ -\infty & \text{for } \Lambda > \ln \lambda_1 \end{cases} \tag{4.7c}$$

$s_h(\Lambda; q)$ is the entropy function for the asymmetrical tent map, given by

$$s_h(\Lambda; q) = [(\Lambda - \ln \lambda_0) \ln |\Lambda - \ln \lambda_0| - (\Lambda - \ln \lambda_1) \ln |\Lambda - \ln \lambda_1|]/\ln(\lambda_0/\lambda_1) + \ln |\ln(\lambda_0/\lambda_1)| - \ln A(q) - q\Lambda \tag{4.8}$$

and $t^*(\Lambda; q)$ is

$$t^*(\Lambda; q) \equiv -q + [\ln(\lambda_0/\lambda_1)]^{-1} \ln |(\Lambda - \ln \lambda_0)/(\Lambda - \ln \lambda_1)| \tag{4.9}$$

Details of the calculation are given in Appendix C. The definitions of Λ_a, Λ_b, and Λ_c are given in that appendix. [See Appendix C for ξ_* and $\xi_0(\Lambda)$ also.]

A minimum point Q of $F(\xi, \eta; \Lambda, q)$ on D tells us the most probable among the reference orbits for which the local Lyapunov exponent is equal to Λ. Indeed, $m = \eta n$ gives the time interval that a reference orbit with an initial value near the fixed point $x = 0$ (or visiting the neighborhood of $x = 0$ at time $T = 1$) stays on the interval $J(0)$. The quantity ξ gives the frequency that a reference orbit visits on the interval $J(1)$. When the last point of a reference orbit passes the neighborhood of $x = 1$, the distance between the last point and $x = 1$ is measured by $i = \zeta n$, i.e., the time necessary for the orbit to pass through the interval $J(0)$ in the subsequent times. For example, let us consider the case $\lambda_0 > \lambda_*$. The minimum point Q is $(\xi, \eta) = (\xi_*, 0)$ for $\Lambda \leqslant \Lambda_a$, where ξ_* is given by (C.2) and not dependent on Λ. A typical orbit with the local Lyapunov exponent Λ has $m \sim O(1)$, $\xi = \xi_*$, and

$$i \sim n[\ln \lambda_0 - \Lambda - \xi_* \ln(\lambda_0/\lambda_1)]/(z-1) \ln \lambda_0$$

The initial point is distant from $x=0$ and $x=1$, the frequency of visiting on $J(1)$ is ξ_*, and the last point passes $x \simeq 1 - \lambda_0^{-zi}$ or λ_0^{-zi}. The probability of such orbits is proportional to the probability of the i state, i.e., $\simeq [1 + (\lambda_0/\lambda_1)^q]^{-i}$. Therefore, $s(\Lambda; q, r = \infty)$ for $\Lambda \leqslant \Lambda_a$ becomes a straight line with the slope $[(z-1) \ln \lambda_0]^{-1} \ln[1 + (\lambda_0/\lambda_1)^q]$.[9] For $\Lambda_a < \Lambda \leqslant \Lambda_b$, $F(\xi, \eta; \Lambda, q)$ on D is minimum at $\xi = (\ln \lambda_0 - \Lambda)/\ln(\lambda_1/\lambda_0)$ and $\eta = 0$. For $\Lambda_b < \Lambda \leqslant z \ln \lambda_0$, the minimum point Q is the intersection (ξ_b, η_b) of the lines $\eta = 1 - \xi/\xi_B$ and $\zeta = 0$. A corresponding orbit takes an initial value around $x \simeq \lambda_0^{-z\eta_b n}$ or $1 - \lambda_0^{-z\eta_b n}$, the frequency of visiting on $J(1)$ is ξ_b, and the last point of the orbit does not fall into the neighborhoods of $x=0$ and $x=1$.

In the fully developed chaos, a chaotic attractor touches the unstable fixed point $x=0$. This contact brings out a singular behavior in the probability distribution of an initial value of a reference orbit. Assume that the probability behaves like $w(x)\, dx \propto x^{\alpha-1}\, dx$ for small x. As Λ_0 denotes a local Lyapunov exponent of the fixed point, the probability that an orbit starts from a point in the interval $[e^{-\Lambda_0(j+1)}, e^{-\Lambda_0 j}]$ is given by $p_j \propto \exp(-\alpha\Lambda_0 j)$. The orbit stays on $J(0)$ till $T = j-1$, and then moves on $J(1)$. I assume that the probability density of the local Lyapunov exponents which are determined by the reference orbits of the part from $T=j$ to $T=n-1$ takes the asymptotic form

$$\exp\{(n-j)\, s_0(\Lambda'; q)\} \qquad \text{for} \quad n-j \gg 1 \tag{4.10}$$

where $s_0(\Lambda'; q)$ is the entropy of the chaotic attractor colliding with the fixed point, which does not include any singularity due to the fixed point; this corresponds to a Markov assumption and holds for the map (2.6) (see Appendix C). The probability that a local Lyapunov exponent takes a value on $[\Lambda, \Lambda + d\Lambda]$ can be written as

$$\begin{aligned} &d\Lambda \sum_{j=1}^{n} P_0 \int_{-\infty}^{\Lambda_M} d\Lambda' \exp\{-\alpha\Lambda_0 j + (n-j)\, s_0(\Lambda'; q)\} \\ &\qquad \times \delta\left(\Lambda - \frac{j}{n}\Lambda_0 - \left(1 - \frac{j}{n}\right)\Lambda'\right) \\ &\simeq d\Lambda \int_{\xi_m}^{1} d\xi\, \frac{nP_0}{1-\xi} \exp\left\{n\left[-\alpha\Lambda_0\xi + (1-\xi)\, s_0\left(\frac{\Lambda - \Lambda_0\xi}{1-\xi}; q\right)\right]\right\} \end{aligned} \tag{4.11}$$

where the normalization factor P_0 may be a power function of j and

$$\xi_m = \begin{cases} 0 & \text{if } \Lambda \leqslant \Lambda_M \\ (\Lambda - \Lambda_M)/(\Lambda_0 - \Lambda_M) & \text{if } \Lambda_M < \Lambda \leqslant \Lambda_0 \end{cases}$$

on the assumption $\Lambda_M < \Lambda_0$. For sufficiently large n, we can use a maximum value approximation to (4.11). Consider the function

$$K(\xi; \Lambda) \equiv -\alpha\Lambda_0 \xi + (1-\xi)\, s_0\left(\frac{\Lambda - \Lambda_0 \xi}{1-\xi}; q\right) \tag{4.12}$$

$K(\xi; \Lambda)$ takes the maximum value K_M at $\xi = \xi^* \equiv (\Lambda - \Lambda^*)/(\Lambda_0 - \Lambda^*)$, where K_M is a linear function of Λ:

$$K_M = [-\alpha(\Lambda - \Lambda^*) + (\Lambda_0 - \Lambda)\, s_0(\Lambda^*; q)]/(\Lambda_0 - \Lambda^*) \tag{4.13a}$$

and Λ^*, a constant independent of Λ, satisfies

$$\frac{\partial s_0(\Lambda^*; q)}{\partial \Lambda^*} = \frac{-\alpha\Lambda_0 - s_0(\Lambda^*; q)}{\Lambda_0 - \Lambda^*} \tag{4.13b}$$

on the assumption of the differentiability of $s_0(\Lambda'; q)$ at $\Lambda' = \Lambda^*$. The condition that $K(\xi; \Lambda)$ takes the maximum value K_M on the interval of integration is $\Lambda^* \leqslant \Lambda \leqslant \Lambda_0$. Note that (4.13) is equal to the equation of the tangent line drawn from the point $(\Lambda_0, -\alpha\Lambda_0)$ to the curve $s = s_0(\Lambda; q)$ on the Λs plane (cf. refs. 4 and 9). Hence, one can say that when a chaotic attractor C collides with a particular fixed point F, the entropy $s = s(\Lambda; q)$ of $C + F$ is given by the convex hull of the entropies of C and F, i.e., $s = s_0(\Lambda; q)$ and the point $(\Lambda_0, -\alpha\Lambda_0)$, respectively.[5,6]

5. DISCUSSION AND SUMMARY

I have developed the statistical thermodynamics of a local Lyapunov exponent on the analogy of the statistical mechanics in thermal equilibrium. Results obtained from the statistical thermodynamics of a local Lyapunov exponent may give important information for understanding the statistical mechanics in thermal equilibrium. Let us consider the Landau free energy[4]

$$\psi(\Lambda, q, t; r = \infty) \equiv \Lambda t - s(\Lambda; q, r = \infty) \tag{5.1}$$

a minimum of which for Λ gives the (Gibbs) free energy $\Delta G(q, t; r = \infty)$. Hereafter, we assume $\lambda_* < \lambda_0 < 2$ and $z > 1$ for simplicity. Figure 9 shows the form of $\psi(\Lambda)$ for $t = t^*(\Lambda_a)$, $t^*(\Lambda_a) < \Lambda < t^*(\Lambda_b)$, and $t = t^*(\Lambda_b)$. In the

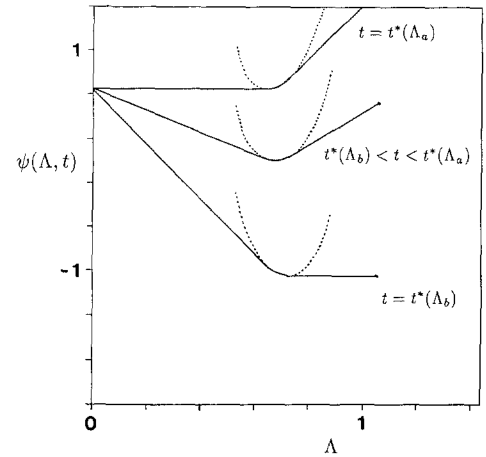


Fig. 9. Landau free energy $\psi(\Lambda, q=1, t; r=\infty)$ for $t=t^*(\Lambda_a)$, $t^*(\Lambda_a)>t>t^*(\Lambda_b)$, and $t=t^*(\Lambda_b)$. Parameter values are $z=2$ and $\lambda_0=1.7$.

thermodynamic limit $n \to \infty$, a unique state of Λ is realized at each t except for $t=t^*(\Lambda_a)$ and $t=t^*(\Lambda_b)$. At $t=t^*(\Lambda_a)$ and $t=t^*(\Lambda_b)$, it occurs that infinitely many states of Λ coexist. Is each of the coexisting states realized with the same probability?[4,25] The probability of the states with $\Lambda \in [\tilde{\Lambda}, \tilde{\Lambda}+d\tilde{\Lambda}]$ at every t is written as

$$d\tilde{\Lambda}\ \tilde{W}_\infty(\tilde{\Lambda}; n, q, t) \equiv d\tilde{\Lambda}\ W_\infty(\tilde{\Lambda}; n, q) \exp(-nt\tilde{\Lambda})\ \Xi_\infty(q, t=0; n)/\Xi_\infty(q, t; n) \tag{5.2}$$

In order to answer the above question, more delicate analyses of the partition function (3.8) are required. That is, we must use second-order approximations of the Stirling formula and steepest decent methods for integration.

At $t=t^*(\Lambda_a)$, the minimum point of $F_D(\xi, \eta; \Lambda, q)$ is independent of Λ for $\Lambda<\Lambda_a$ (see Appendix C). Since $F(\xi, \eta; \Lambda, q)$ is a linear function of Λ whose coefficient is independent of ξ and η, all of the coexisting states have the same probability. For the coexistence at $t=t^*(\Lambda_b)$, direct calculations of (5.2) are given in Appendix D. Here I give an intuitive discussion. Let us use (4.11) for an evaluation of the probability $\tilde{W}$. Then, we must determine an asymptotic form of the j dependence of P_0. The normalization condition of the probability (4.10) leads to

$$P_0 \sim [2\pi(n-j)\ |s_0''(\Lambda_\infty; q)|]^{1/2} \tag{5.3}$$

where Λ_∞ is the Lyapunov exponent and s_0'' denotes the second derivative of $s_0(\Lambda; q)$ for Λ. Expanded in Taylor series about the maximum point $\xi = \xi_*$, $K(\xi; \Lambda)$ can be written as

$$K(\xi; \Lambda) = K_M + \frac{(\Lambda_0 - \Lambda^*)^2}{2(1-\xi_*)} s_0''(\Lambda^*; q)(\xi - \xi_*)^2 + \cdots \tag{5.4}$$

for $\Lambda^* < \Lambda < \Lambda_0$. Inserting (5.3) and (5.4) into (4.11) and using steepest decent methods, we have

$$W_\infty(\Lambda; n, q)\, d\Lambda = \frac{nC}{\Lambda_0 - \Lambda^*} e^{nK_M} \left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\} d\Lambda \qquad \text{for} \quad \Lambda^* < \Lambda < \Lambda_0$$

where C is a constant. Since the Landau free energy is constant for $\Lambda^* = \Lambda_b < \Lambda < \Lambda_0 = z \ln \lambda_0$ at $t = t^*(\Lambda_b)$, we have

$$\begin{aligned} &\tilde{W}_\infty(\Lambda; n \to \infty, q, t^*(\Lambda_b)) \\ &\quad = \begin{cases} d\Lambda/(\Lambda_b - z \ln \lambda_0) & \text{for} \quad \Lambda_b < \Lambda < z \ln \lambda_0 \\ 0 & \text{for} \quad \Lambda < \Lambda_b \quad \text{or} \quad \Lambda > z \ln \lambda_0 \end{cases} \end{aligned} \tag{5.5}$$

Remark that the same minimum value of a Landau free energy does not imply that coexisting states have the same probability of realization (cf. refs. 4 and 25).

The thermodynamics of a local Lyapunov exponent has been studied on a nonhyperbolic attractor of maps conjugate to an asymmetrical tent map. The free energy has been obtained exactly and shows qualitatively different behaviors, depending on the asymmetry of a map. The free energy has been generalized by using weighted measures for an initial value of a reference orbit so that a relation between the thermodynamics of a scaling index and of a local Lyapunov exponent, especially the occurrence of first-order phase transitions, has become clear. The entropy function has been directly calculated from the partition function, since the Legendre transform of the free energy may be different from the entropy function. Symbolic dynamics can explicitly write a relation between a local Lyapunov exponent and reference orbit. Typical reference orbits have been given for each value of a local Lyapunov exponent. Linear slopes of the entropy have been explained in intuitive discussions. We found that the collision of a chaotic attractor with a particular fixed point yields a singular local structure in the distribution of a local Lyapunov exponent. It has been shown that each of the coexisting states at the phase transition points is realized with the same probability in the thermodynamic limit.

APPENDIX A. CALCULATION OF THE PARTITION FUNCTION

Exchanging the order of summation, we can write the partition function (3.8) as

$$\begin{aligned}\Xi_N(q, t; n) = &\sum_{m=2}^{n} (1/\lambda_0)^{(q+t)m} (\lambda_0/\lambda_1)^{q+t} \sum_{\sigma_{m+1}\cdots\sigma_n} \left\{ \prod_{j=m+1}^{n} [1/\lambda(\sigma_j)]^{q+t} \right\} \\ &\times \sum_{\sigma_{n+1}\cdots\sigma_N} \left\{ \prod_{j=n+1}^{N} [1/\lambda(\sigma_j)]^{q} \right\} \{ |\varphi'(\tilde{u}(0\cdots 01\sigma_{m+1}\cdots\sigma_N))|^t \\ &+ (\lambda_0/\lambda_1)^{q+t} |\varphi'(\tilde{u}(10\cdots 01\sigma_{m+1}\cdots\sigma_N))|^t \} \\ &\times |\varphi'(\tilde{u}(\sigma_{n+1}\cdots\sigma_N))|^{-t} \end{aligned} \tag{A.1}$$

Assume that there exist positive constants K_i ($i = 1,..., 4$) of $O(1)$ such that

$$K_1 u^{z-1} < \varphi'(u) < K_2 u^{z-1} \qquad \text{for} \quad 0 < u < a \tag{A.2a}$$

$$K_3(1-u)^{z-1} < \varphi'(u) < K_4(1-u)^{z-1} \qquad \text{for} \quad a < u < 1 \tag{A.2b}$$

We denote the inequality (A.2a) as $\varphi'(u) \propto u^{z-1}$. From (2.5) and (2.8b), the following inequalities hold:

$$\begin{aligned} &u(0\cdots 0\overset{j}{\check{1}}\,10\cdots 0) = \lambda_0^{-j} < \tilde{u}(0\cdots 01\sigma_{j+1}\cdots\sigma_N) < \lambda_0^{-(j-1)} \\ &u(10\cdots 0\overset{j}{\check{1}}\,0\cdots 0) = 1 - (\lambda_0/\lambda_1)\,\lambda_0^{-(j-1)} \\ &\qquad < \tilde{u}(10\cdots 01\sigma_{j+1}\cdots\sigma_N) < 1 - (\lambda_0/\lambda_1)\,\lambda_0^{-j} \end{aligned} \tag{A.3}$$

Substituting (A.3) into (A.2), we have

$$\begin{aligned} &\varphi'(\tilde{u}(0\cdots 01\sigma_{j+1}\cdots\sigma_N)) \propto \lambda_0^{-(z-1)(j-1)} \\ &\varphi'(\tilde{u}(10\cdots 01\sigma_{j+1}\cdots\sigma_N)) \propto (\lambda_0/\lambda_1)^{z-1}\,\lambda_0^{-(z-1)(j-1)} \end{aligned} \tag{A.4}$$

Therefore, (A.1) can be written as

$$\begin{aligned} \Xi_N(q, t; n) \propto &\left[1 + \left(\frac{\lambda_0}{\lambda_1}\right)^{q+zt}\right] \sum_{m=2}^{n} \left(\frac{1}{\lambda_0}\right)^{(q+zt)(m-1)} [A(q+t)]^{n-m} \\ &\times \sum_{\sigma_{n+1}\cdots\sigma_N} \prod_{j=n+1}^{N} \left[\frac{1}{\lambda(\sigma_j)}\right]^q |\varphi'(\tilde{u}(\sigma_{n+1}\cdots\sigma_N))|^{-t} \\ \propto &\left[1 + \left(\frac{\lambda_0}{\lambda_1}\right)^{q+zt}\right] \sum_{m=2}^{n} \left(\frac{1}{\lambda_0}\right)^{(q+zt)(m-1)} [A(q+t)]^{n-m} \\ &\times \left[1 + \left(\frac{\lambda_0}{\lambda_1}\right)^{q-(z-1)t}\right] \sum_{i=2}^{N-n} \left(\frac{1}{\lambda_0}\right)^{[q-(z-1)t](i-1)} [A(q)]^{N-n-i} \end{aligned} \tag{A.5}$$

The summations in (A.5) lead to (3.9).

APPENDIX B. ANALYSIS OF $G(q, t; r>1)$

We study a solution of Eq. (3.15)

$$(z-1)t\ln\lambda_0+\ln[1+(\lambda_0/\lambda_1)^{q+t}]=0$$

At first, consider the case $z>1$,

$$Q(t)\equiv -t+[\ln(\lambda_0-1)]^{-1}\ln|\lambda_0^{-(z-1)t}-1| \tag{B.1}$$

A solution of (3.15) satisfies $q=Q(t)$. Now, $Q(t)$ is a continuous function for $t<0$, monotone decreasing from $+\infty$ to $-\infty$ if $\lambda_0>\lambda_1$, and monotone increasing from $-\infty$ to $+\infty$ if $\lambda_1>\lambda_0>\lambda_*$, where λ_* is a maximum real solution of the equation

$$\lambda_*^{z-1}(\lambda_*-1)=1 \qquad (\lambda_*>1) \tag{B.2}$$

Therefore, we have

$$t_*=Q^{-1}(q)<0 \qquad \text{for} \quad \lambda_0>\lambda_* \tag{B.3}$$

For $t<t_*$, we have

$$(z-1)t\ln\lambda_0+\ln[1+(\lambda_0/\lambda_1)^{q+t}]<0 \tag{B.4}$$

When $\lambda_*>\lambda_0>1$, $Q(t)$ tends to positive infinity as $t\to -0$ and $t\to -\infty$. The concave function $Q(t)$ for $t<0$ has the minimum q_c at $t=t_c$,

$$t_c\equiv[(z-1)\ln\lambda_0]^{-1}\ln|B|<0 \tag{B.5a}$$

$$q_c\equiv\frac{1}{(z-1)\ln\lambda_0}\left[\frac{(z-1)\ln\lambda_0}{\ln(\lambda_0-1)}\ln\left|\frac{(z-1)\ln\lambda_0}{\ln(\lambda_0-1)}\right|-B\ln|B|\right] \tag{B.5b}$$

where $B\equiv[\ln(\lambda_0-1)]^{-1}\ln|\lambda_0^{z-1}(\lambda_0-1)|$. If $q<q_c$, (3.15) has no real solution and there is no region of the phase C. If $q>q_c$, (3.15) has two real solutions, denoted by t_m and t_M ($t_m<t_c<t_M<0$). For $q>0$, t_M coincides with $t_*=Q^{-1}(q)$ at $\lambda_0=\lambda_*$ in the limit $\lambda_0\to\lambda_*-0$. Indeed, we have

$$t_M=t^*-\frac{(q+t^*)\lambda_*(\lambda_*-1)^{q-1}+(z-1)t^*}{\lambda_*[1-(\lambda_*-1)^q]\ln(\lambda_*-1)}(\lambda_*-\lambda_0)+O((\lambda_*-\lambda_0)^2)$$

$$t_m=\frac{q\lambda_*(\lambda_*-1)\ln(\lambda_*-1)}{z\lambda_*+1-z}\frac{1}{\lambda_*-\lambda_0}+O(1)$$

for $0<\lambda_*-\lambda_0\ll 1$ if $q>q_c$, where

$$t^*\equiv Q^{-1}(q;\lambda_0=\lambda_*)=-[\ln(\lambda_*-1)]^{-1}\ln|1-(\lambda_*-1)^q|$$

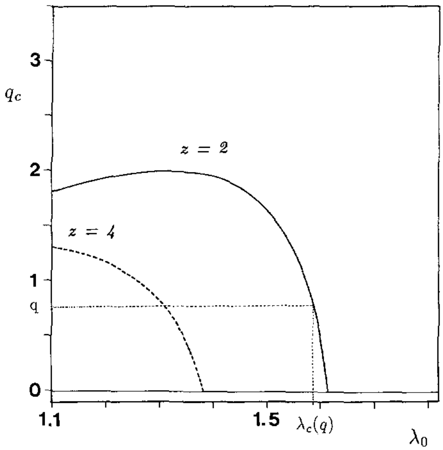


Fig. 10. Plot of $q_c = q_c(\lambda_0)$, given by (B.5b). The heavy line is for $z = 2$ and the dashed line for $z = 4$. When $\lambda_* > \lambda_0 > 1$, the free energy $G(q, t; r = 1)$ for $q > q_c$ has two phase transitions from B to C and from C to B as t decreases.

Figure 10 gives q_c in (B.5b). For small $q > 0$, there is a value of λ_0 which satisfies $q_c(\lambda_0) = q$ for $1 < \lambda_0 < \lambda_*$, denoted by $\lambda_c(q)$. For $q \leqslant 0$, $\lambda_c(q)$ is defined as $\lambda_c(q) = \lambda_*$. If $1 < \lambda_0 < \lambda_c(q)$, then we have

$$(z-1)t \ln \lambda_0 + \ln[1 + (\lambda_0/\lambda_1)^{q+t}] > 0$$

for every t. If $\lambda_c(q) < \lambda_0 < \lambda_*$, the phase C is realized for $t_m < t < t_M$. At $z = 2$ and $q = 1$, we have $\lambda_* = \frac{1}{2}(1 + \sqrt{5}) = 1.61803...$, $\lambda_c = 1.57745...$, and $t^* = -2$. Let us consider the case $0 < z < 1$. The solution of (3.15) is positive. $Q(t)$ is given for $t > 0$ by (B.1). Using the same arguments, we get that the phase C is realized for $t > t_* > 0$ if $1 < \lambda_0 < \lambda_*$, and for $0 < t_m < t < t_M$ if $\lambda_* < \lambda_0 < \lambda_c(q)$, while there is no region of the phase C if $\lambda_0 > \lambda_c(q)$. [Note that $\lambda_c(q) = \lambda_*$ for $q \geqslant 0$.]

APPENDIX C. CALCULATION OF $s(\Lambda; q, r = \infty)$

$F(\xi, \eta; \Lambda, q)$ is defined on the region

$$D_0 \equiv \{(\xi, \eta) \mid 0 \leqslant \xi \leqslant 1 - \eta \text{ and } 0 \leqslant \eta \leqslant 1\}$$

by (4.6), where we put $x \ln x = 0$ at $x = 0$. We write

$$D_1 \equiv \{(\xi, \eta) \mid (z-1)\eta \ln \lambda_0 + \ln \lambda_0 - \xi \ln(\lambda_0/\lambda_1) - \Lambda \geqslant 0\}$$

The region of integration in (4.5) is $D \equiv D(\Lambda; q, r=\infty) = D_0 \cap D_1$. Let $F_D(\xi, \eta; \Lambda, q)$ denote the restriction of $F(\xi, \eta; \Lambda, q)$ to D. We study a minimum point and minimum value of $F_D(\xi, \eta; \Lambda, q)$ for $z > 1$.

Proposition C.1. $F(\xi, \eta; \Lambda, q)$ is a function of class C^∞ on the interior of D, convex and monotone increasing for η, and convex for ξ.

Proposition C.2. The surface $\tau = F(\xi, \eta; \Lambda, q)$ in the $\xi\eta\tau$ plane is a ruled surface.

(ξ_0, η_0) is an inner point of D_0. The surface $\tau = F(\xi, \eta; \Lambda, q)$ has the principal curvatures 0 in the direction $(-\xi_0, 1-\eta_0)$ and $[(1-\eta_0)^2 + \xi_0^2]/\xi_0(1-\eta_0)(1-\eta_0-\xi_0)$ in the direction $(\xi_0, 1-\eta_0)$ at (ξ_0, η_0). Proposition C.1 yields that $F(\xi, \eta; \Lambda, q)$ is a minimum at a point on the line $\eta = 0$.

Lemma C.3. At $(\xi, \eta) = (\xi_*, 0)$, $F(\xi, \eta; \Lambda, q)$ takes the minimum value

$$F(\xi_*, 0; \Lambda, q) = (\Lambda - z \ln \lambda_0)(q - \theta) - \ln[1 + (\lambda_0/\lambda_1)^\theta] \tag{C.1}$$

where θ and ξ_* are given by

$$\theta \equiv q + [(z-1)\ln \lambda_0]^{-1} \ln[1 + (\lambda_0/\lambda_1)^q] \tag{C.2a}$$

$$\xi_* \equiv 1/[1 + (\lambda_1/\lambda_0)^\theta] \tag{C.2b}$$

Corollary C.4. For $\Lambda \leqslant \Lambda_a \equiv \ln \lambda_0 - \xi_* \ln(\lambda_0/\lambda_1)$, $F_D(\xi, \eta; \Lambda, q)$ takes the minimum value (C.1) at $(\xi, \eta) = (\xi_*, 0)$.

If $\Lambda > \Lambda_a$, then $(\xi_*, 0) \notin D$. From Proposition C.1, it follows that a minimum point of F_D lies on the lines

$$\partial D_0 \equiv \{(\xi, \eta) \mid \eta = 0 \text{ and } (\xi, \eta) \in D\} \tag{C.3a}$$

and

$$\partial D_1 \equiv \{(\xi, \eta) \mid (z-1)\eta \ln \lambda_0 + \ln \lambda_0 - \xi \ln(\lambda_0/\lambda_1) - \Lambda = 0 \text{ and } (\xi, \eta) \in D\} \tag{C.3b}$$

We investigate the behavior of $F(\xi, \eta; \Lambda, q)$ on a line $\zeta = \text{const}$. Assume $\lambda_0 \neq \lambda_1$. Then, we define

$$H(\xi, \eta; \Lambda, q) \equiv \frac{\partial F}{\partial \eta} + \frac{(z-1)\ln \lambda_0}{\ln(\lambda_0/\lambda_1)} \frac{\partial F}{\partial \xi} \tag{C.4}$$

Since $F(\xi, \eta; \Lambda, q)$ is concave on ∂D_1, the following holds:

Proposition C.5. $H(\xi, \eta; \Lambda, q)$ monotonically increases on ∂D_1 w.r.t. η.

Λ_m and Λ_M are the minimum and maximum values of $\{\ln \lambda_0, \ln \lambda_1\}$, respectively. Putting

$$\xi_0 \equiv \xi_0(\Lambda) = (\ln \lambda_0 - \Lambda)/\ln(\lambda_0/\lambda_1)$$

we have $(\xi_0, 0) \in \partial D_0 \cap \partial D_1$ for $\Lambda_m < \Lambda < \Lambda_M$. Let t_* denote the maximum real solution of (3.15), i.e., $t_* = Q^{-1}(q)$ for $\lambda_0 \geqslant \lambda_*$ and $t_* = t_M$ for $\lambda_* > \lambda_0 > \lambda_c(q)$. We define

$$\Lambda_b \equiv \ln \lambda_1 + \lambda_0^{(z-1)t_*} \ln(\lambda_0/\lambda_1) \qquad \text{for} \quad \lambda_0 > \lambda_c(q) \tag{C.5a}$$

$$\Lambda_c \equiv \ln \lambda_1 + \lambda_0^{(z-1)t_m} \ln(\lambda_0/\lambda_1) \qquad \text{for} \quad \lambda_* > \lambda_0 > \lambda_c(q) \tag{C.5b}$$

Proposition C.6. For $q > 0$ and $\lambda_* > \lambda_0 > \lambda_c(q)$,

$$\Lambda_a < \Lambda_b < z \ln \lambda_0 < \Lambda_c \tag{C.6a}$$

$$\xi_0(\Lambda_b) < \left| \frac{(z-1)\ln \lambda_0}{\ln(\lambda_0/\lambda_1)} \right| < \xi_0(\Lambda_c) \tag{C.6b}$$

Inserting (C.2) into the definition of Λ_a and using (3.15) in (C.5a), we have

$$\Lambda_b - \Lambda_a = \{[1 + (\lambda_0/\lambda_1)^{q+t_*}]^{-1} - [1 + (\lambda_0/\lambda_1)^{\theta}]^{-1}\} \ln(\lambda_0/\lambda_1)$$

Since $t_* < 0 < \theta - q$, we have $\Lambda_b > \Lambda_a$. From (B.5a), it turns out that

$$z \ln \lambda_0 = \ln \lambda_1 + \lambda_0^{(z-1)t_c} \ln(\lambda_0/\lambda_1)$$

Therefore, we get (C.6a). Using

$$\xi_0(\Lambda_b) = 1 - \lambda_0^{(z-1)t_*} < 1 - \lambda_0^{(z-1)t_c} < \xi_0(\Lambda_c) = 1 - \lambda_0^{(z-1)t_m}$$

we get (C.6b). ∎

Proposition C.7. For $\lambda_0 > \lambda_*$, $H(\xi, \eta; \Lambda, q)$ vanishes on the line $\eta = 1 - \xi/\xi_0(\Lambda_b)$. For $\lambda_* > \lambda_0 > \lambda_c$, $H(\xi, \eta; \Lambda, q)$ vanishes on the lines $\eta = 1 - \xi/\xi_0(\Lambda_b)$ and $\eta = 1 - \xi/\xi_0(\Lambda_c)$.

Actually, we have

$$\begin{aligned} H\left(\xi, \eta = 1 - \frac{\xi}{\xi_0(\Lambda)}; \Lambda, q\right) &= -\ln[1 - \xi_0(\Lambda)] + \frac{(z-1)\ln \lambda_0}{\ln(\lambda_0/\lambda_1)} \ln \frac{\xi_0(\Lambda)}{1 - \xi_0(\Lambda)} - q(z-1)\ln \lambda_0 \\ &= -(z-1)\, t_* \ln \lambda_0 + \frac{(z-1)\ln \lambda_0}{\ln(\lambda_0/\lambda_1)} \\ &\quad \times \ln(\lambda_0^{-(z-1)t_*} - 1) - q(z-1)\ln \lambda_0 \qquad \text{at} \quad \Lambda = \Lambda_b \end{aligned} \tag{C.7}$$

Using (3.15), we get $H(\xi, \eta = 1 - \xi/\xi_0(\Lambda_b); \Lambda_b, q) = 0$. Since $t = t_m$ is a solution of (3.15) for $\lambda_* > \lambda_0 > \lambda_c$, the same argument leads to $H(\xi, \eta = 1 - \xi/\xi_0(\Lambda_c); \Lambda_c, q) = 0$. ▮

Lemma C.8. For $\Lambda_a < \Lambda < \Lambda_b$, if $\lambda_0 > \lambda_*$, $\Lambda_a < \Lambda \leqslant \Lambda_b$ or $\Lambda_c < \Lambda \leqslant \ln \lambda_1$ if $\lambda_* \geqslant \lambda_0 > \lambda_c$, and $\Lambda_a < \Lambda \leqslant \ln \lambda_1$ if $\lambda_c > \lambda_0 > 1$, $F_D(\xi, \eta; \Lambda, q)$ takes the minimum value $-s_h(\Lambda; q)$ at $(\xi, \eta) = (\xi_0, 0)$, where $s_h(\Lambda; q)$ is given by (4.8).

For $\lambda_0 > \lambda_*$ and $\Lambda_a < \Lambda < \Lambda_M$, $H(\xi_0, 0; \Lambda, q)$ monotonically decreases w.r.t. Λ [see (C.7)] and vanishes at $\Lambda = \Lambda_b$. It turns out from Proposition C.5 that for $\Lambda_a < \Lambda \leqslant \Lambda_b$ $(< \Lambda_M)$, $F_D(\xi, \eta; \Lambda, q)$ is a minimum at $(\xi, \eta) = (\xi_0, 0)$. If $\lambda_* > \lambda_0 > 1$, $H(\xi_0, 0; \Lambda, q)$ monotonically decreases on the interval $(\Lambda_a, z \ln \lambda_0)$ and monotonically increases on the interval $(z \ln \lambda_0, \ln \lambda_1)$. The minimum value at $\Lambda = z \ln \lambda_0$ can be written as

$$H(\xi_0, 0; \Lambda = z \ln \lambda_0, q) = (q_c - q)(z - 1) \ln \lambda_0$$

where q_c is given by (B.5b). It is trivial that $H(\xi_0, 0; \Lambda = z \ln \lambda_0, q) > 0$ if $q \leqslant 0$. When $q > 0$ and $\lambda_c > \lambda_0 > 1$, $H(\xi_0, 0; \Lambda = z \ln \lambda_0, q) > 0$ because $q > q_c$. Therefore, it turns out from Proposition C.5 that for $\lambda_c > \lambda_0 > 1$, $F_D(\xi, \eta; \Lambda, q)$ is a minimum at $(\xi, \eta) = (\xi_0, 0)$. When $q > 0$ and $\lambda_* \geqslant \lambda_0 > \lambda_c$, $H(\xi_0, 0; \Lambda = z \ln \lambda_0, q) < 0$. Proposition C.8 gives $H(\xi_0, 0; \Lambda = \Lambda_b, q) = H(\xi_0, 0; \Lambda = \Lambda_c, q) = 0$. From (C.6a), it follows that $H(\xi_0, 0; \Lambda, q) \geqslant 0$ for $\Lambda_a < \Lambda \leqslant \Lambda_b$ or $\Lambda_c \leqslant \Lambda < \ln \lambda_1$. A straightforward calculation of $F(\xi_0, 0; \Lambda, q)$ gives the minimum value $-s_h(\Lambda; q)$. ▮

Lemma C.9. Assume that $q > 0$ and $\lambda_* > \lambda_0 > \lambda_c$. Then, $F_D(\xi, \eta; \Lambda, q)$ is a minimum at the intersection (ξ_b, η_b) of the lines ∂D_1 and $\eta = 1 - \xi/\xi_0(\Lambda_b)$ if $\Lambda_b < \Lambda < z \ln \lambda_0$, and at the intersection (ξ_c, η_c) of the lines ∂D_1 and $\eta = 1 - \xi/\xi_0(\Lambda_c)$ if $z \ln \lambda_0 < \Lambda < \Lambda_c$. The minimum values are

$$F(\xi_b, \eta_b; \Lambda, q) = -(\Lambda - z \ln \lambda_0)\, t_* + q \ln \lambda_0 + \ln A(q) \quad \text{for} \quad \Lambda_b < \Lambda < z \ln \lambda_0 \tag{C.8a}$$

$$F(\xi_c, \eta_c; \Lambda, q) = -(\Lambda - z \ln \lambda_0)\, t_m + q \ln \lambda_0 + \ln A(q) \quad \text{for} \quad z \ln \lambda_0 < \Lambda < \Lambda_c \tag{C.8b}$$

From Proposition C.6, it follows that ∂D_1 intersects the line $\eta = 1 - \xi/\xi_0(\Lambda_b)$ if $\Lambda_b < \Lambda < z \ln \lambda_0$, and the line $\eta = 1 - \xi/\xi_0(\Lambda_c)$ if $z \ln \lambda_0 < \Lambda < \Lambda_c$. Since $H(\xi, \eta; \Lambda, q)$ vanishes at these intersections (from Proposition C.7), it turns out from Propositions C.5 and C.1 that the intersections are minimum points of F_D. Inserting $\eta = 1 - \xi/\xi_0(\Lambda_b)$ into (4.6) yields

$$F\left(\xi, \eta = 1 - \frac{\xi}{\xi_0(\Lambda_b)}; \Lambda, q\right)$$

$$= \xi[(z-1)\, t_* \ln \lambda_0 - q \ln \lambda_0 - \ln A(q)] \left[\frac{1}{\xi_0(\Lambda_b)} + \frac{\ln(\lambda_0/\lambda_1)}{(z-1)\ln \lambda_0}\right]$$

$$+ [q \ln \lambda_0 + \ln A(q)] \left[2 + \frac{1}{z-1} - \frac{\Lambda}{(z-1)\ln \lambda_0}\right]$$

Using

$$\xi_b[(z-1)\ln \lambda_0/\xi_0(\Lambda_b) + \ln(\lambda_0/\lambda_1)] = z \ln \lambda_0 - \Lambda$$

in the above at $(\xi, \eta) = (\xi_b, \eta_b)$, we obtain (C.8a). The same argument leads to (C.8b). ▌

Since a maximum value approximation of the integral (4.5) is valid for large n, $s(\Lambda; q, r=\infty)$ is equal to a maximum value of $-F_D(\xi, \eta; \Lambda, q)$. Inserting Λ_a into (4.8) and (4.9), we have

$$t^*(\Lambda_a; q) = \theta - q \tag{C.9a}$$

$$\begin{aligned} s_h(\Lambda_a; q) &= \ln[1 + (\lambda_0/\lambda_1)^\theta] - \theta \ln \lambda_0 + (\theta - q)\,\Lambda_a - \ln A(q) \\ &= \ln[1 + (\lambda_0/\lambda_1)^\theta] + (\Lambda_a - z \ln \lambda_0)(\theta - q) \end{aligned} \tag{C.9b}$$

Inserting Λ_b into (4.8) and (4.9) and using (3.15), we have

$$t^*(\Lambda_b; q) = t_* \tag{C.10a}$$

$$\begin{aligned} s_h(\Lambda_b; q) &= -(z-1)\, t_* \ln \lambda_0 - (q + t_*)[1 - \lambda_0^{(z-1)t_*}] \\ &\quad \times \ln(\lambda_0/\lambda_1) - \ln A(q) - q\Lambda_b \\ &= t_*(\Lambda_b - z \ln \lambda_0) - q \ln \lambda_0 - \ln A(q) \end{aligned} \tag{C.10b}$$

The same calculations for $t^*(\Lambda_c; q)$ and $s_h(\Lambda_c; q)$ yield

$$t^*(\Lambda_c; q) = t_m \tag{C.11a}$$

$$s_h(\Lambda_c; q) = t_m(\Lambda_c - z \ln \lambda_0) - q \ln \lambda_0 - \ln A(q) \tag{C.11b}$$

Using Corollary C.4, Lemmas C.8 and C.9, and Eqs. (C.9)–(C.11), we get (4.7).

APPENDIX D. CALCULATION OF THE PROBABILITY (5.2)

The partition function (3.8) can be written from (4.3) as

$$\Xi_N(q, t; n) = \Xi_N(q, t=0; n) \int d\Lambda\, W_N(\Lambda; n, q)\, e^{-nt\Lambda} \tag{D.1}$$

Inserting (A.5) into (D.1), using second-order approximations of the Stirling formula, and replacing the summations with integrations, we get

$$W_\infty(\Lambda; n, q)\, e^{-nt\Lambda} \sim n^{5/2} \iint_D d\xi\, d\eta \left[\frac{1-\eta}{(1-\eta-\xi)\xi}\right]^{1/2} \times \exp[-nF(\xi, \eta; \Lambda, q) - nt\Lambda] \tag{D.2}$$

We introduce new variables (u, v) by

$$\begin{aligned} \eta - 1 &= -(\xi - u)/\xi_0(\Lambda_b) \\ \Lambda &= (z-1)(\eta - v)\ln\lambda_0 + \ln\lambda_0 - \xi\ln(\lambda_0/\lambda_1) \end{aligned} \tag{D.3}$$

and write

$$F^*(u, v; \Lambda, q) \equiv F(\xi, \eta; \Lambda, q) \tag{D.4}$$

As we assume $\lambda_* < \lambda_0 < 2$ and $z > 1$, $F_D^*(u, v; \Lambda, q)$ for $\Lambda_b < \Lambda < z\ln\lambda_0$ is a minimum at $(u, v) = (0, 0)$;

$$\frac{\partial F_D^*}{\partial u}(u=0, v; \Lambda, q) = 0, \qquad \frac{\partial F_D^*}{\partial v}(u=0, v; \Lambda, q) = C_1 > 0 \tag{D.5a}$$

$$\frac{\partial^2 F_D^*}{\partial u^2}(u=0, v; \Lambda, q) = \frac{1}{\xi_0^*[1-\xi_0(\Lambda_b)]} > 0 \tag{D.5b}$$

where C_1 is a positive constant, not dependent on Λ and v, and $\xi_0^* \equiv \xi(u=0, v)$ is a linear function of Λ and v. Performing the integration of u by steepest decent methods, we have

$$W_\infty(\Lambda; n, q)\, e^{-nt\Lambda} = C_2 n \exp\{n[s_h(\Lambda_b; q) - \Lambda_b t^*(\Lambda_b; q)]\} \times \exp\{-n\Lambda[t - t^*(\Lambda_b; q)]\}\left\{1 + O\left(\frac{1}{\sqrt{n}}\right)\right\} \tag{D.6}$$

for $\Lambda_b < \Lambda < z\ln\lambda_0$. Here C_2 is a positive constant independent of Λ, t, and n. At $t = t^*(\Lambda_b; q)$, it follows that

$$\tilde{W}_\infty(\Lambda; n, q, t)\, d\Lambda = \begin{cases} d\Lambda/(z\ln\lambda_0 - \Lambda_b) & \text{for} \quad \Lambda_b < \Lambda < z\ln\lambda_0 \\ 0 & \text{for} \quad \Lambda_b > \Lambda \quad \text{or} \quad \Lambda > z\ln\lambda_0 \end{cases} \tag{D.7}$$

in the thermodynamic limit $n \to \infty$.

ACKNOWLEDGMENTS

I am grateful to Prof. H. Fujisaka and Prof. H. Mori for their valuable discussions and encouragement. I also thank Dr. T. Yoshida, Dr. T. Horita, Dr. H. Hata, and the other members of the research group on chaos at Kyushu University for many useful discussions.

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